

A new Gross-Pitaevskii action for cold Fermi condensates

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The BEC regime of a cold fermi gas is characterised by coupled atoms (dimers) which, superficially, look very much like elementary bosons. We construct a new Gross-Pitaevskii action for the BEC regime in which dimers are represented by coupled fields, which enables us to determine how simply bosonic they really are. To exemplify this we construct vortex solutions. We find that there is a difference, which although very small in the deep BEC regime, becomes significant when the BCS regime is approached.

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INTRODUCTION

Condensates of ultra-cold fermi gases are richer in their properties than those of elementary bosons, interpolating from a BCS regime characterised by Cooper pairs to a BEC regime characterised by diatoms or dimers. Within this range of behaviour dimers look closest to elementary bosons and the main aim of the paper is to see if, and to what extent, they differ from them. We do this through the Gross-Pitaevskii (GP) equations, whose description of elementary bosons is wholly familiar.

We have already seen [1] that, in the BEC regime, cold fermi gases obey a simple Gross-Pitaevskii equation in the acoustic (or hydrodynamic) approximation. The main result of this paper is to provide a generalised Gross-Pitaevskii description of fermi gases with greater validity. Unsurprisingly, in the BEC regime we shall show that, as strongly coupled atoms, dimers are represented by a pair of Gross-Pitaevskii fields, whose dynamics is described by strongly coupled Gross-Pitaevskii actions. In fact, our overall action is, formally, an extension of the generalised GP action derived many years ago by Aitchison, Thouless *et al.* [2] for fermi gases in the BCS regime of Cooper pairs. This is despite the difference between their model and ours, in that our gas is controlled through an explicit Feshbach resonance and theirs not, and that we are considering the opposite regime of strong rather than weak coupling.

As a demonstration of this generalised action we look for vortex solutions. The existence of vortices in cold fermi gases is confirmed experimentally [3–5] and there has been extensive theoretical analysis of them in the BCS regime [6], the unitary limit [7, 8], and throughout the BCS-BEC crossover [9, 10], based upon Bogoliubov-de Gennes theory [11, 12]. In particular, these studies indicate that the effective Gross-Pitaevskii description for composite bosons can be provided only in strong coupling of the BEC regime with which, from our very different viewpoint, we agree. In practice, on comparing the vortex equations for condensates both of elementary bosons and of fermi gas composite bosons we find near-identity of the solutions despite the different equations that each satisfy. This helps confirm the supposition that the strongly-coupled dimers of the BEC regime do behave like elementary bosons to a good approximation. At the same time, we can see how this approximate identity breaks down as we move away from the deep regime.

THE MODEL

Our model describes a condensate of fermi atoms ψ_σ with spin $\sigma = (\uparrow, \downarrow)$, mass m , interacting through an idealised narrow Feshbach resonance $\varphi(x)$ [13]. All calculations are performed at temperature $T = 0$ where, in the absence of

external fields, the action is taken to be (in units in which $\hbar = 1$) [13]

$$\begin{aligned}
S = & \int dt d^3x \left\{ \sum_{\uparrow, \downarrow} \psi_{\sigma}^*(x) \left[i \partial_t + \frac{\nabla^2}{2m} + \mu \right] \psi_{\sigma}(x) \right. \\
& + \varphi^*(x) \left[i \partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \varphi(x) \\
& \left. - g [\varphi^*(x) \psi_{\downarrow}(x) \psi_{\uparrow}(x) + \varphi(x) \psi_{\uparrow}^*(x) \psi_{\downarrow}^*(x)] - \frac{1}{4} u_B |\varphi(x)|^4 \right\}, \tag{1}
\end{aligned}$$

where we have extended the single-channel model of [13] to include self-interactions (the final term in (1)) which, in the BEC regime, can be interpreted as dimer-dimer interactions along the lines of [14, 15]. The mass of the dimer bosons is $M = 2m$, and ν is the binding energy of the resonance which is tuned by an external magnetic field. It is not difficult to accommodate direct s-wave interactions [16, 17] in (1) but it complicates the formalism hugely even though it leads to some unchanged outcomes in e.g. the behaviour of the speed of sound in the deep BEC and BCS regimes. We have examined the model of [13] without such self-interactions $u_B = 0$ in some detail in previous papers [1, 18, 19], from which we borrow where appropriate.

Since the action is quadratic in the Fermi fields, they can be integrated out to give the exact (one fermi-loop) bosonic action

$$S_{NL} = -i \text{Tr} \ln \mathcal{G}^{-1} + \int dt d^3x \left\{ \varphi^*(x) \left[i \partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \varphi(x) - \frac{1}{4} u_B |\varphi(x)|^4 \right\}, \tag{2}$$

with \mathcal{G}^{-1} the fermi loop inverse Nambu Green function,

$$\mathcal{G}^{-1} = \begin{pmatrix} i\partial_t - \varepsilon & -g\varphi(x) \\ -g\varphi^*(x) & i\partial_t + \varepsilon \end{pmatrix}. \tag{3}$$

Here $\varepsilon = -\nabla^2/2m - \mu$ (or in momentum space, $\varepsilon_k = k^2/2m - \mu$) is the energy measured from the Fermi surface.

The condensate of the theory (with a minus sign for convenience) is

$$\varphi(x) = -|\varphi(x)| e^{i\theta(x)} \tag{4}$$

with the gapless mode encoded in the phase θ . Subsequently we adopt the mean-field approximation, with greater justification for narrow resonances [13], and neglect fluctuations about the extrema of S_{NL} .

The action (2) is invariant under global $U(1)$ transformations $\theta \rightarrow \theta + c$ for some constant c but the variational equation $\delta S_{NL} = 0$ permits constant solutions $\varphi(x) = \varphi_0$ which break this symmetry spontaneously, satisfying

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{1}{2E_p} - \frac{1}{2\varepsilon_p} \right) = \frac{\nu - 2\mu}{g^2} + \frac{u_B}{2g^2} |\varphi_0|^2 \equiv -\frac{m}{4\pi a_S} \tag{5}$$

where $E_p = (\varepsilon_p^2 + g^2|\phi_0|^2)^{1/2}$. The second equation in (5) defines the atomic s-wave scattering length a_S . Moreover, the fermion number density is

$$\rho_0 = \rho_0^F + \rho_0^B, \tag{6}$$

where $\rho_0^F = \int d^3\mathbf{p}/(2\pi)^3 [1 - \varepsilon_p/E_p]$ is the explicit fermion density, and $\rho_0^B = 2|\phi_0|^2$ is due to molecules (two fermions per molecule).

We now make a Galilean-invariant decomposition

$$\varphi(x) = -(|\varphi_0| + \delta|\varphi(x)|) e^{i\theta(x)} \tag{7}$$

as a precursor to a Galilean-invariant expansion of S_{NL} in terms of the derivatives of θ and the *small* fluctuation in the condensate density $\delta|\varphi| = |\varphi| - |\varphi_0|$ and its derivatives. The Galilean invariants of the theory are the density fluctuation $\delta|\varphi|$ itself, $G(\theta) = \dot{\theta} + (\nabla\theta)^2/2M$, and $D_t(\delta|\varphi|, \theta) = \partial_t(\delta|\varphi|) + \nabla\theta \cdot \nabla(\delta|\varphi|)/M$. D_t is the comoving time derivative of $\delta|\varphi|$ in a fluid with fluid velocity $\nabla\theta/M$. $\theta(x)$ is not small.

On changing variables to θ and the dimensionless $\epsilon = \kappa^{-1/2} \delta|\varphi|$, so that $S[\varphi, \varphi^*] \rightarrow S[\theta, \epsilon]$ we expand $S_{NL}[\theta, \epsilon]$ in powers of the Galilean invariants along the lines taken in [2] and sketched in [1, 16–19]. All the terms relevant to

particle identification are contained in the *local* Galilean invariant effective density [1, 18, 19]

$$S_{eff}[\theta, \epsilon] = \int d^4x \left[\frac{N_0}{4} G^2(\theta, \epsilon) - \frac{1}{2} \rho_0 G(\theta, \epsilon) - \alpha \epsilon G(\theta, \epsilon) + \frac{1}{4} \eta D_t^2(\epsilon, \theta) - \frac{1}{4} \bar{M}^2 \epsilon^2 \right]. \quad (8)$$

The scale factor κ has been chosen so that on extending $G(\theta)$ to $G(\theta, \epsilon) = \dot{\theta} + (\nabla\theta)^2/2M + (\nabla\epsilon)^2/2M$, ϵ has the same coefficients as θ in its spatial derivatives [2]. We can write \bar{M}^2 as

$$\bar{M}^2 = \bar{M}_0^2 + 6u_B\kappa|\varphi_0|^2, \quad (9)$$

where the second term arises explicitly from the self-interaction. The term linear in ϵ , which corresponds to making the replacement $\alpha G \rightarrow \alpha G + u_B\kappa^{1/2}|\varphi|^3$ in (8) has no effect on the form of (8), since it always contributes to total derivatives in the calculations which follow. It only serves to modify the gap equation in (5). For the moment we shall postpone the inclusion of the terms $O(\epsilon^3)$ and $O(\epsilon^4)$ in (8) that follow from the interaction term $u_B|\varphi(x)|^4$. In practice the effects of these terms are negligible in the BEC limit and they will be introduced only to confirm their insignificance. We note that these terms are equally negligible in the deep BCS regime considered in [2], where $|\varphi_0|$ vanishes.

To identify the mode spectrum we retain the quadratic part of $S_{eff}[\theta, \epsilon]$,

$$S_{qu}[\theta, \epsilon] = \int d^4x L_{qu}(\theta, \epsilon) \quad (10)$$

where

$$L_{qu}(\theta, \epsilon) = \frac{1}{4} \left\{ N_0 \dot{\theta}^2 - \frac{\rho_0}{M} (\nabla\theta)^2 - 2\alpha(\epsilon\dot{\theta} - \dot{\epsilon}\theta) + \eta\dot{\epsilon}^2 - \frac{\rho_0}{M} (\nabla\epsilon)^2 - \bar{M}^2 \epsilon^2 \right\} \quad (11)$$

and we have anti-symmetrised the time derivatives. The equations of motion following from (11) are

$$\begin{aligned} N_0 \ddot{\theta} - \frac{\rho_0}{M} \nabla^2 \theta - 2\alpha \dot{\epsilon} &= 0 \\ \eta \ddot{\epsilon} - \frac{\rho_0}{M} \nabla^2 \epsilon + \bar{M}^2 \epsilon + 2\alpha \dot{\theta} &= 0. \end{aligned} \quad (12)$$

For plane waves solutions straightforward calculation gives the phonon dispersion relation

$$\omega^2 = c^2 k^2 + c_1^2 k^4 + \dots \quad (13)$$

where the speed of sound c takes the form

$$c^2 = \frac{\rho_0}{MN}. \quad (14)$$

in which

$$N \equiv N_0 + \frac{4\alpha^2}{M^2} \quad (15)$$

Applying an external magnetic field enables us to change the binding energy and hence, from (5), the s-wave scattering length a_S . Conveniently, a_S^{-1} is approximately linear in the applied field [13]. It is convenient to measure all parameters in units of $1/a_S k_F$, where k_F is the Fermi momentum. Negative $1/a_S k_F$ corresponds to the BCS regime, positive $1/a_S k_F$ to the BEC regime. $1/a_S k_F = 0$ defines the *unitary* limit of crossover phenomena. A word on terminology: by *deep* regimes we mean large $1/k_F a_S$ regimes, corresponding to large- ν regimes.

In the Appendix we give the algebraic definitions of the coefficients of the terms in (11) together with some exemplary figures for typical gases, taking self-interaction $u_B \neq 0$ into account (see also [19]). For a given scattering length a_S , solving the first equality in (5) together with the conservation of the fermion density (6) determines the chemical potential μ and the spacetime independent condensate field $|\phi_0|$, both independent of the dimer-dimer interaction u_B . Then, from the second equality of Eq.(5), the tunable threshold energy of Feshbach resonances ν can be determined

in the presence of u_B in terms of a_S . Thus, $N_0, \eta, \alpha, \kappa$ do not depend on the value of u_B (see Figs.2). However, \bar{M}^2 does depend on u_B directly, leading to a dependence on u_B of c^2 (see Fig.3).

The key attributes of the coefficients relevant to the BEC regime following from Figs.2 are that i) at positive values of $1/a_S k_F$ (the BEC regime) the mode-coupling coefficient $\alpha \rightarrow \rho_0$ ii) both N_0 and $\eta \approx N_0 \rightarrow 0$, with $N_0/\eta \rightarrow 1$. [On the other hand, for comparison with the model of [2], in the deep BCS regime $\eta = N_0$ is the density of states at the Fermi surface and $\alpha \rightarrow 0$.] iii) \bar{M}_0^2 also vanishes and, relevant for the definition of \bar{M}^2 , iv) $\kappa^{-1}|\varphi_0|^2 \rightarrow 1$.

We note the very different behaviours of N_0 and N in the deep BEC regime. Whereas N_0 vanishes there N does not. In particular, in the idealised model (8) of [13] with no dimer self-interactions $u_B = 0$ the vanishing of \bar{M}^2 in the BEC limit forces $c \rightarrow 0$ because of the *divergence* of N . The extension of the model [13] to include a direct s-wave atomic interaction in (1), which leads to a complicated two-fluid picture [16–18], still leaves $c \rightarrow 0$ in the deep BEC regime [16]. We shall not consider this extension here. However, the inclusion of direct dimer-dimer interactions $u_B \neq 0$ as a consequence of $\varphi(x)$ -field interactions leads to non-vanishing speed of sound in the deep BEC region. Defining $\bar{M}_B^2 = 6u_B|\varphi_0|^2$ from (9) then, with $\bar{M} \rightarrow \bar{M}_B$ and $\alpha \rightarrow \rho_0$, $c^2 \rightarrow c_{BEC}^2 = \bar{M}_B^2/4M\rho_0$.

As we might have anticipated, $c_1^2 = O(1/M^4)$ as we approach the BEC regime and, indeed, equals $(1/2M)^2$ in the BEC limit but we shall not pursue the non-linearity further except to say that the fluctuation field ϵ now provides the quantum pressure normally understood as leading to the Bogoliubov non-linearity.

In addition to the gapless phonon dispersion relation there is another (gapped) solution to (12) which, in the BEC regime $N_0 \approx \eta \approx 0$, takes the form

$$\omega^2 \approx \frac{4\alpha^2}{N_0^2} + \frac{2\rho_0}{MN_0}k^2 + \dots \quad (16)$$

As $N_0 \rightarrow 0$ this gap, related to the ϵ field, diverges. Naively, this suggests that the ϵ field ceases to be a dynamical field and drops out of the picture. We shall see that this is true in part, but that a residue of the gapped mode survives in the equations of motion.

GROSS-PITAEVSKII EQUATIONS

Before attempting to construct Gross-Pitaevskii equations which provide a realisation of (8) we make the simplest approximation (see [2]) in which we are only concerned with the phase fluctuations of φ and set $\epsilon = 0$. As before, since the Gross-Pitaevskii field describes diatoms of mass $M = 2m$ we express masses in terms of M rather than m .

The Lagrangian density of (8) then reduces to

$$L_{\text{eff}}^0(\theta) = -\frac{1}{2}\rho_0 G(\theta) + \frac{N_0}{4}G^2(\theta). \quad (17)$$

in terms of the Galilean scalar $G(\theta) = \dot{\theta} + (\vec{\nabla}\theta)^2/2M$ which describes a single non-relativistic degree of freedom θ . The Euler-Lagrange equation for θ can be rewritten as a continuity equation of a *single* fluid,

$$\frac{\partial}{\partial t}\rho + \vec{\nabla} \cdot (\rho \mathbf{v}) = 0, \quad (18)$$

with $\mathbf{v} = \vec{\nabla}\theta/M$, where

$$\rho = \rho_0 - N_0 G(\theta). \quad (19)$$

On rearrangement [2] Eq.(19) is Bernoulli's equation, demonstrating the hydrodynamic nature of the action (17). To obtain the dispersion relation, it is sufficient to expand the Lagrangian density (17) to second order,

$$L_{\text{qu}}^0(\theta) = \frac{N_0}{4} \left[\dot{\theta}^2 - \frac{\rho_0}{N_0 M} (\vec{\nabla}\theta)^2 \right] \quad (20)$$

The dispersion relation $\omega^2 = c_0^2 k^2$ follows directly, where

$$c_0^2 = \frac{\rho_0}{N_0 M}. \quad (21)$$

defines the speed of sound c_0 .

Now consider the Gross-Pitaevskii (GP) Lagrangian density describing the wave-function ψ of a particle of mass M , interacting non-linearly with itself as

$$L_{GP}^0(\psi) = i\psi^*\dot{\psi} - \frac{1}{2M}\nabla\psi^* \cdot \nabla\psi - \frac{1}{2N_0}(|\psi|^2 - \rho_0)^2. \quad (22)$$

The superfix zero on the definitions of (17), (20) and (22) is a reminder that $L^0(\theta)$ and $L_{GP}^0(\psi)$ are defined in terms of N_0 and c_0 . On adopting the Madelung phase-modulus decomposition $\psi = \sqrt{\rho}e^{i\theta}$, L_{GP}^0 becomes

$$L_{GP}^0(\psi) \equiv L_{GP}^0(\rho, \theta) = -\rho\dot{\theta} - \frac{1}{8M}\frac{(\vec{\nabla}\rho)^2}{\rho} - \frac{\rho}{4M}(\vec{\nabla}\theta)^2 - \frac{1}{2N_0}(\rho - \rho_0)^2. \quad (23)$$

Varying with respect to θ reproduces (18) on using the same definition for \mathbf{v} . Varying with respect to ρ gives

$$\rho - \rho_0 + N_0 G(\theta) - \frac{1}{4M}\vec{\nabla} \cdot \left(\frac{\vec{\nabla}\rho}{\rho} \right) = 0. \quad (24)$$

Neglecting the final term on the left hand side of the equation (the hydrodynamic approximation) recreates the Bernoulli Eq.(19), showing the identity between the different formalisms.

The approximation of setting $\epsilon = 0$ shown above is a very poor one in the BEC regime because of the vanishing of N_0 or, equivalently, the divergence of c_0^2 there. A better approximation is if, rather than setting $\epsilon = 0$ in (8), we neglect the spatial and temporal variation of ϵ in comparison to ϵ itself. Then (8) reduces to

$$L_{\text{eff}}(\theta, \epsilon) = \frac{N_0}{4}G^2(\theta) - \frac{1}{2}\rho_0 G(\theta) - \alpha\epsilon G(\theta) - \frac{1}{4}\bar{M}^2\epsilon^2 \quad (25)$$

and the ϵ Euler-Lagrange equation gives $\epsilon \approx -2\alpha G(\theta)/\bar{M}^2$, a slave to the phase. In this generalised hydrodynamic approximation [18] (25) becomes

$$L_{\text{eff}}^{\text{hyd}}[\theta] = -\frac{1}{2}\rho_0 G(\theta) + \frac{N}{4}G^2(\theta), \quad (26)$$

the Lagrangian density (17) for elementary bosons but with N_0 replaced by N of (15).

This extension preserves the relationship between the form (8) and the simple Gross-Pitaevskii equation and all the results of the previous subsection go through identically provided we replace N_0 with N everywhere. That is, in the hydrodynamic limit an inverse Madelung transformation gives the Lagrangian density

$$L_{GP}^{\text{hyd}}(\psi) = i\psi^*\dot{\psi} - \frac{1}{2M}\nabla\psi^* \cdot \nabla\psi - \frac{1}{2N}(|\psi|^2 - \rho_0)^2, \quad (27)$$

more familiar as

$$L_{GP}^{\text{hyd}}(\psi) = i\psi^*\dot{\psi} - \frac{1}{2M}\nabla\psi^* \cdot \nabla\psi - \frac{Mc^2}{2\rho_0}(|\psi|^2 - \rho_0)^2, \quad (28)$$

where the phonon propagates no longer with speed c_0 given in (21) but c given in (14). That is, the hydrodynamic approximation has, as (28), the dual role of also being the density for a condensate of *elementary* bosons. In the context of the Gross-Pitaevskii framework the extent to which dimers are not elementary bosons is measured by the distance between the generalised GP Lagrangian density that we shall derive and (28).

We shall return to (25) later, but for the more general action (8) we follow the tactics used in [2] in describing a BCS model of cold fermi gases. [The paper [2] keeps the derivatives of ϵ but sets $\alpha = 0$, as is the case for the deep BCS regime.] As with [2] we retain $\eta = N_0$, as appropriate for the deep BEC regime as it is for the deep BCS. The BEC Lagrangian density now becomes

$$L_{\text{eff}}(\theta, \epsilon) = \frac{N_0}{4}G^2(\theta, \epsilon) - \frac{1}{2}\rho_0 G(\theta, \epsilon) - \alpha\epsilon G(\theta, \epsilon) + \frac{N_0}{4}D_t^2(\epsilon, \theta) - \frac{1}{4}\bar{M}^2\epsilon^2. \quad (29)$$

Again as in [2], we work with the combinations of modes

$$\theta = \frac{\theta_+ + \theta_-}{2}, \quad \epsilon = \frac{\theta_+ - \theta_-}{2}, \quad (30)$$

whence

$$G(\theta, \epsilon) = \frac{1}{2}[G(\theta_+) + G(\theta_-)] \quad (31)$$

and

$$\left(\dot{\epsilon} + \frac{\nabla \epsilon \nabla \theta}{M} \right) = \frac{1}{2}[G(\theta_+) - G(\theta_-)]. \quad (32)$$

In terms of θ_{\pm} , $L_{\text{eff}}(\theta, \epsilon)$ becomes

$$2L_{\text{eff}}(\theta_+, \theta_-) = [L_{\text{eff}}^0(\theta_+) + L_{\text{eff}}^0(\theta_-)] - \frac{1}{2}\bar{M}^2 \left(\frac{\theta_+ - \theta_-}{2} \right)^2 - \alpha(G(\theta_+) + G(\theta_-)) \left(\frac{\theta_+ - \theta_-}{2} \right) \quad (33)$$

where $L_{\text{eff}}^0(\theta)$ is defined in (17) in terms of N_0 and c_0 .

If we define

$$\rho_{\pm} = \rho_0 - N_0 G(\theta_{\pm}) + \alpha(\theta_+ - \theta_-), \quad (34)$$

$$\mathbf{v}_{\pm} = \nabla \theta_{\pm} / M, \quad (35)$$

then varying with respect to θ_{\pm} gives the equations of motion

$$\partial_t \rho_{\pm} + \vec{\nabla}(\rho_{\pm} \mathbf{v}_{\pm}) = \pm \left\{ \frac{\bar{M}^2 N}{2N_0} (\theta_+ - \theta_-) + \frac{\alpha}{N_0} [2\rho_0 - (\rho_+ + \rho_-)] \right\}, \quad (36)$$

representing continuity equations with sources and sinks.

We want to reproduce these from a GP action, necessarily with two pseudo-fields

$$\psi_{\pm} = \sqrt{\rho_{\pm}} e^{i\theta_{\pm}} \quad (37)$$

where ρ_{\pm} and θ_{\pm} are considered independent. A partial inverse Madelung transformation gives the Lagrangian density (written in mixed formalism)

$$L_{GP}(\psi_+, \psi_-) = [L_{GP}^0(\psi_+) + L_{GP}^0(\psi_-)] - \frac{\bar{M}^2}{4N_0} N(\theta_+ - \theta_-)^2 - \frac{\alpha}{N_0} [2\rho_0 - (\rho_+ + \rho_-)] (\theta_+ - \theta_-) \quad (38)$$

where $L_{GP}^0(\psi)$ is defined in (22) in terms of N_0 and c_0 . Varying with respect to ρ_{\pm} and θ_{\pm} gives precisely the equations (34) and (36) at relevant order. To convert the interaction Lagrangian density fully to GP fields, we express θ_{\pm} and ρ_{\pm} in terms of ψ_{\pm} as

$$(\theta_+ - \theta_-) = \frac{1}{2i} \ln \left(\frac{\psi_+ \psi_-^*}{\psi_- \psi_+^*} \right), \quad (39)$$

$$\rho_{\pm} = \psi_{\pm}^* \psi_{\pm}. \quad (40)$$

On substituting

$$\psi_{\pm} = \sqrt{\rho_0} + \Psi_{\pm} \quad (41)$$

we obtain the dispersion relation for long wavelength modes $\omega^2 = c^2 k^2 + \dots$ with $c^2 = \rho_0 / MN$ of (14), as required.

We had not included the further powers in ϵ arising from the self-interaction $u_B |\varphi|^4 / 4$ in anticipation that they were unimportant in the BEC regime, as we shall see later. Their inclusion is straightforward. Eq.(33) contains the term $\bar{M}^2 (\theta_+ - \theta_-)^2 / 16 = \bar{M}^2 \epsilon^2 / 4 = (\bar{M}_0^2 + 6u_B |\varphi_0|^4) (\epsilon^2 / 4)$ where we have taken the BEC limit $\kappa = |\varphi_0|^2$ to write $u_B |\varphi|^4 / 4 = u_B |\varphi_0|^4 (1 + \epsilon)^4 / 4 = u_B |\varphi_0|^4 (1 + 4\epsilon + 6\epsilon^2 + 4\epsilon^3 + \epsilon^4) / 4$ for simplicity. The terms of order ϵ^0 and ϵ^1 have already been taken into account, and (33) takes into account the term $O(\epsilon^2)$. The complete Lagrangian density is, again in mixed notation,

$$\begin{aligned} L_{GP}(\psi_+, \psi_-) &= [L_{GP}^0(\psi_+) + L_{GP}^0(\psi_-)] - \frac{\bar{M}^2}{4N_0} N(\theta_+ - \theta_-)^2 - \frac{\alpha}{N_0} [2\rho_0 - (\rho_+ + \rho_-)] (\theta_+ - \theta_-) \\ &\quad - \frac{1}{8} u_B \rho_0^2 (\theta_+ - \theta_-)^3 - \frac{1}{32} u_B \rho_0^2 (\theta_+ - \theta_-)^4. \end{aligned} \quad (42)$$

where we have used $2|\varphi_0|^2 = \rho_0$ in the BEC regime.

Entirely in terms of the fields ψ_{\pm} this becomes

$$\begin{aligned} L_{GP}(\psi_+, \psi_-) = & [L_{GP}^0(\psi_+) + L_{GP}^0(\psi_-)] + \frac{\bar{M}^2}{16N_0} N \left[\ln \left(\frac{\psi_+ \psi_-^*}{\psi_- \psi_+^*} \right) \right]^2 - \frac{i\alpha}{2N_0} [\psi_+^* \psi_+ + \psi_-^* \psi_- - 2\rho_0] \ln \left(\frac{\psi_+ \psi_-^*}{\psi_- \psi_+^*} \right) \\ & - \frac{i u_B \rho_0^2}{64} \left[\ln \left(\frac{\psi_+ \psi_-^*}{\psi_- \psi_+^*} \right) \right]^3 - \frac{u_B \rho_0^2}{256} \left[\ln \left(\frac{\psi_+ \psi_-^*}{\psi_- \psi_+^*} \right) \right]^4. \end{aligned} \quad (43)$$

Eq.(43) is the main result of this paper, an effective GP theory in the vicinity of the deep BEC regime. We note that, if we insert the deep BCS parameter values $\alpha, |\varphi_0| \rightarrow 0$ in (43) we recover the action of [2] despite the replacement of the Feshbach resonance by an s-wave point atom-atom interaction in their calculations. However, we cannot extend Eq.(43) to the intermediate regime e.g. in the vicinity of the unitary regime, because the form (43) is predicated on the equality of N_0 and η , which breaks down there, as can be seen in the Appendix. Away from the BEC regime we have to fall back upon (8), with the additional terms $O(\epsilon^3)$ and $O(\epsilon^4)$ omitted there requiring inclusion.

Even for the BEC regime, using the Lagrangian density (43) to determine condensate behaviour looks more difficult than working with (8) directly. In particular, we note that $L_{GP}(\psi_+, \psi_-)$ looks to be singular in the deep BEC regime where $N_0 \rightarrow 0$ (and c_0 becomes arbitrarily large). [Remember that $L_{GP}(\psi_+)$ and $L_{GP}(\psi_-)$ are also defined in terms of N_0 and c_0 .] However, a positive consequence of this singular behaviour is that the final two terms in (43) become insignificant in the BEC regime because they are relatively $O(N_0)$, making the model more tractable. That is, the effect of the dimer quartic self-interaction with $u_B \neq 0$ is only to give a non-zero speed of sound in the BEC limit. Having said that, we need to demonstrate that, in practice, there is no singular behaviour, as we shall now see.

STATIC VORTEX SOLUTIONS

For all the caveats of the previous paragraph, Gross-Pitaevskii theory provides the best way to understand vortices. We have already explored vortex production in the context of the hydrodynamic approximation [1] and we return to the problem with the more sophisticated Lagrangian density of Eq.(43). Previous studies of vortices in this superfluid system were based upon Bogoliubov-de Gennes theory [6–10], whose coarse-graining permits a *time-independent* equation of the gap parameter, from which the static single vortex state can be constructed [11, 12]. These studies suggest that the effective Gross-Pitaevskii description for composite bosons can be provided only in strong coupling of the BEC regime, which we will see agrees with our formalism above and our earlier work [1].

We saw that the BEC fermi gas condensate behaved like a condensate of elementary bosons in the hydrodynamic approximation of (25). Such condensates show vortices with global $U(1)$ vortex solutions that are well-understood [20]. Our more realistic system shown above in (43), without this approximation, also permits vortex solutions and we shall contrast them to those of the simpler system. We shall see that there are no experimentally observable differences in the vortex profile.

With single vortices in mind, it is sufficient to construct the static limit of the Gross-Pitaevskii actions. At the level of approximation that we shall adopt here we insert the deep BEC values $\alpha = \rho_0$ and $N = N_0 + 4\rho_0^2/\bar{M}^2$, but do not set $N_0 \rightarrow 0$ until calculations have been performed, because of its singular nature.

The static Hamiltonian H_{GP} following from the Lagrangian density L_{GP} of (43) is then

$$\begin{aligned} H_{GP}(\psi_+, \psi_-) = & H_{GP}^0(\psi_+) + H_{GP}^0(\psi_-) \\ & - \int d^3x \left\{ \frac{\rho_0}{2iN_0} [\psi_+^* \psi_+ + \psi_-^* \psi_- - 2\rho_0] \ln \left(\frac{\psi_+ \psi_-^*}{\psi_- \psi_+^*} \right) \right. \\ & \left. + \frac{\bar{M}^2}{16N_0} N \left[\ln \left(\frac{\psi_+ \psi_-^*}{\psi_- \psi_+^*} \right) \right]^2 \right\}, \end{aligned} \quad (44)$$

where

$$H_{GP}^0(\psi) = \int d^3x \left\{ -\psi^* \left[\frac{1}{2M} \vec{\nabla}^2 + \frac{\rho_0}{N_0} \right] \psi + \frac{1}{2N_0} (|\psi|^2)^2 \right\}, \quad (45)$$

the Hamiltonian following from the Lagrangian density $L_{GP}^0(\psi)$ of (23). For clarity of equations, initially we have neglected the terms in $(\theta_+ - \theta_-)^3$ and $(\theta_+ - \theta_-)^4$. We shall include them later.

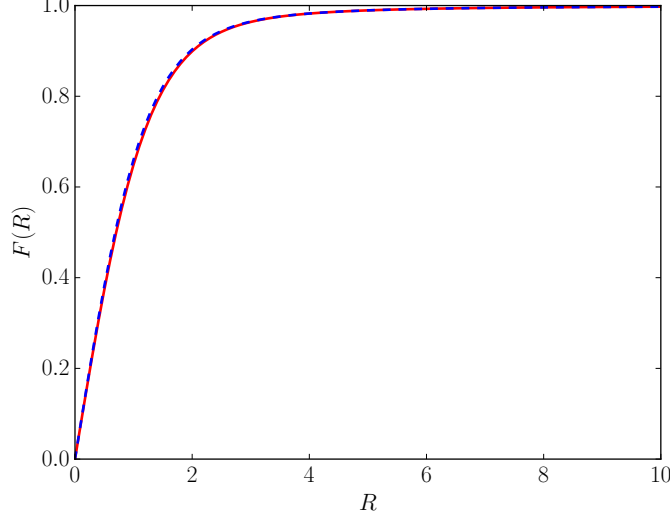


FIG. 1: Numerical solution (blue profile) of the BEC vortex Eq.(56) compared to the solution to the global U(1) vortex Eq.(60) for a condensate of elementary bosons (red profile). The difference is insignificant.

To see how vortices arise in (44), in cylindrical polars (r, z, φ) we make the simple ansatz

$$\psi_{\pm}(r, \varphi) = f(r)e^{\pm i\epsilon(r)}e^{i\varphi} \quad (46)$$

where $f^2 = \rho$. H_{GP} then takes the form (up to an additive constant)

$$\begin{aligned} H_{GP}(f, \epsilon) = & 2\pi \int r dr \left\{ \frac{1}{M} \left[f'^2 + \epsilon'^2 f^2 + \frac{1}{r^2} f^2 \right] r + \frac{1}{N_0} [\rho_0 - f^2]^2 \right. \\ & \left. + \frac{4\rho_0}{N_0} [\rho_0 - f^2] \epsilon + \frac{\bar{M}^2}{N_0} N \epsilon^2 \right\}. \end{aligned} \quad (47)$$

The resulting equations of motion are

$$0 = f'' + \frac{f'}{r} - \epsilon'^2 f - \frac{1}{r^2} f + 2f \frac{M}{N_0} [\rho_0 - f^2] + \frac{4M\rho_0}{N_0} f \epsilon \quad (48)$$

$$0 = \left[\epsilon'' + \frac{1}{r} \epsilon' \right] f^2 - \frac{2M\rho_0}{N_0} [\rho_0 - f^2] - \frac{M\bar{M}^2}{N_0} N \epsilon \quad (49)$$

The limit of vanishing N_0 in these equations is subtle. We develop iterative solutions in powers of N_0 before taking the limit. The effect is to solve (49) as

$$\epsilon \approx -\frac{2\rho_0}{\bar{M}^2 N} (\rho_0 - f^2), \quad (50)$$

where the approximation reflects the deep BEC limit, and then insert this result into (48). This is a consequence of the vanishing of N_0 , which in turn can be understood as an acoustic approximation of neglecting derivatives of ϵ , with its divergent gap in the dispersion relation (16) when $N_0 \rightarrow 0$. As a result

$$\epsilon' \approx \frac{4\rho_0 f f'}{\bar{M}^2 N} = \frac{f f'}{\rho_0} \quad (51)$$

in the small N_0 limit. Inserting (50) and (51) in (48) gives the modified vortex equation

$$0 = f'' + \frac{f'}{r} - f'^2 \frac{f^3}{\rho_0^2} - \frac{1}{r^2} f + \frac{2M^2 c^2}{\rho_0} f [\rho_0 - f^2] \quad (52)$$

After some intricate cancellation the singular terms in $1/N_0$ have been replaced by benign factors of $1/N$ or, equivalently, $c^2 = \rho_0/MN$. Although, from (50), there is a core of density fluctuations, the energy within this cannot be separated from that due to the density profile in (47), again because of the cancelations that are required to give a finite answer. It is because of the difficulties of implementing these cancelations that other seemingly plausible ansatz for vortices, such as trying to put all the vortex structure into just one of the pseudo-fields (e.g. taking $\psi_- = \rho^{1/2}$) fail.

Adding the terms in ϵ^3 and ϵ^4 necessary to get a non-zero c^2 in the BEC regime has no effect on the final equations. To see this we observe that the inclusion of these terms replaces H_{GP} of (47) by

$$H_{GP}(f, \epsilon) = 2\pi \int r dr \left\{ -\frac{1}{M} \left[f f'' - \epsilon'^2 f^2 + \frac{1}{r} f f' - \frac{1}{r^2} f^2 \right] - 2f^2 \frac{\rho_0}{N_0} + \frac{f^4}{N_0} \right. \quad (53)$$

$$\left. + \frac{4\rho_0}{N_0} [\rho_0 - f^2] \epsilon + \frac{\bar{M}^2}{N_0} N \epsilon^2 + u_B \rho_0^2 \epsilon^3 + \frac{u_B}{4} \rho_0^2 \epsilon^4 \right\}. \quad (54)$$

Thus (49) becomes

$$0 = \left[\epsilon'' + \frac{1}{r} \epsilon' \right] f^2 - \frac{2M\rho_0}{N_0} [\rho_0 - f^2] - \frac{M\bar{M}^2}{N_0} N \epsilon - 3Mu_B \rho_0^2 \epsilon^2 - Mu_B \rho_0^2 \epsilon^3 \quad (55)$$

On multiplying through by N_0 , we see that the additional terms do not contribute as N_0 vanishes and (50), (51) and (52) persist.

In particular, the extraction of the bosonic condensate from the fermi pairs does not introduce any new length scales. The equation is cast in dimensionless form by rewriting f as $f \equiv \sqrt{\rho_0} F$, in units of length $r = \xi R$, where $\xi = \hbar/Mc$ is the length scale of the model. On taking the $N_0 \rightarrow 0$ limit into account Eq.(52) then becomes

$$0 = F''(R) + \frac{F'(R)}{R} - F'(R)^2 F(R)^3 - \frac{1}{R^2} F(R) + 2F(R)[1 - F(R)^2] \quad (56)$$

where the primes denote differentiation with respect to R and we have used the fact that $\alpha \approx \rho_0$ in the BEC regime. The boundary conditions are $F(0) = 0, F(\infty) = 1$, whence $F(R) \sim R$, small R , as for the simple $U(1)$ global vortex displayed below. The numerical solution is given in Fig.1.

We contrast this result with that obtained from the static Hamiltonian H_{GP}^{hyd} , the Hamiltonian for elementary bosons, derived from the Lagrangian density (23) after replacing N_0 by N ,

$$H_{GP}^{\text{hyd}}(\psi) = \int d^3x \left\{ -\psi^* \left[\frac{1}{2M} \vec{\nabla}^2 + \frac{\rho_0}{N} \right] \psi + \frac{1}{2N} (|\psi|^2)^2 \right\} \quad (57)$$

where ψ is the single GP field in this approximation.

The ansatz

$$\psi(r, \varphi) = f(r) e^{i\varphi} \quad (58)$$

leads to the more familiar $U(1)$ vortex equation [20],

$$0 = f'' + \frac{f'}{r} - \frac{1}{r^2} f + \frac{2M^2 c^2}{\rho_0} f [\rho_0 - f^2]. \quad (59)$$

In identical dimensionless units the equation for the conventional global $U(1)$ vortex is [20]

$$0 = F''(R) + \frac{F'(R)}{R} - \frac{1}{R^2} F(R) + 2F(R)[1 - F(R)^2] \quad (60)$$

The difference between (59) and the diatomic (52) lies entirely in the $F'(R)^2 F(R)^3$ term, of universal strength in the deep BEC regime. The effect of the linear behaviour near the origin is to render the contribution from the $F'(R)^2 F(R)^3$ term negligible, as is seen in the numerical solutions to Eqs. (56) and (60) of Fig.1, all but indistinguishable to the naked eye. That is, as far as the vortex profile is concerned it is sufficient to work with the simple GP action of (25) for elementary bosons.

Some caution is required in interpreting these results. The original formulation of (8) assumed small fluctuations ϵ . Although convenient factors of N_0 enabled higher powers of ϵ to occur without having to invoke the smallness of the fluctuations, we are forced to address this in our BEC solution (50), which becomes $\epsilon \approx -(1 - F^2)/2$ whereas, by definition, $\epsilon \rightarrow -(1 - F)$ in the BEC limit. Thus, technically, we should only believe the vortex solution from (56) away from the vortex core where $F \approx 1$. In practice, since F has to vanish at the core, we anticipate the vortex solution of Fig.1 to have greater validity.

CONCLUSIONS

The main goal of this work was to understand to what extent diatoms/dimers in the BEC regime of a cold fermi gas resembled elementary bosons. To see this we have developed a generalised Gross-Pitaevskii action for two coupled fields which describes the diatom dynamics. However, this construction is singular in that the individual field actions and the coupling between them are couched in terms of an anomalous speed of sound c_0 which diverges in the deep BEC regime. In practice divergent terms cancel to give a behaviour couched in terms of the real speed of sound c . A by-product of this behaviour is to give the model greater applicability (e.g. to accommodate dimer-dimer interactions) that might have been anticipated. We have shown this in the calculation of vortex equations, contrasting those of the BEC fermi gas to those of elementary bosons or, equivalently, the fermi gas in the hydrodynamic approximation. Although universally different they turn out to be almost indistinguishable numerically, when they are valid. This near-identity of the solutions is the second result of this paper and helps confirm the supposition that the strongly-coupled dimers of the BEC regime do behave like elementary bosons to a good approximation.

As we move away from the deep BEC region towards the BCS regime the situation becomes more complicated. Firstly, the hydrodynamic approximation (23) itself breaks down somewhere in the vicinity of the transition [1] but the generalised action (43) remains valid. For vortex solutions we can proceed as before, with equations of motion (48) and (55) with $N_0 \neq \eta$ and ρ_0 replaced by α where appropriate (see Fig.2). With N_0 no longer very small the deviation of the resulting vortex profiles from those of elementary bosons is no longer universal, and we have not attempted any solutions.

We stress that the effective *time-dependent* Gross-Pitaevskii theory in the BEC regime can, in principle, be implemented beyond the static approximation to study the dynamical aspects of vortices, something that is difficult from the Bogoliubov- de Gennes perspective.

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APPENDIX

In this appendix, we provide the detailed expression of the coefficients of each of terms in the effective action (11). The explicit fermi density is given by

$$\rho_0^F = \int d^3\mathbf{p}/(2\pi)^3 [1 - \varepsilon_p/E_p] \quad (61)$$

where, in conventional notation, $\varepsilon_k = k^2/2m$ and $E_p = (\varepsilon_p^2 + g^2|\phi_0|^2)^{1/2}$. In the presence of direct dimer-dimer interactions ($u_B \neq 0$) they take the form (after renormalisation [16])

$$N_0 = g^2|\phi_0|^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p^3}, \quad (62)$$

$$\alpha = 2|\phi_0|\kappa^{1/2} \left[1 + \frac{1}{2}g^2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\varepsilon_p}{2E_p^3} \right], \quad (63)$$

$$\bar{M}^2 = \bar{M}_0^2 + 6\kappa u_B |\phi_0|^2; \quad \bar{M}_0^2 = -\frac{g^2\kappa m}{\pi a_S} - 2g^2\kappa \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[\frac{\varepsilon_p^2}{E_p^3} - \frac{1}{(\mathbf{p}^2/2m)} \right], \quad (64)$$

$$\eta = g^2\kappa \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\varepsilon_p^2}{2E_p^5}. \quad (65)$$

The scale factor κ is defined as

$$\kappa = \frac{\rho_0}{4mg^2\zeta + 2}, \quad (66)$$

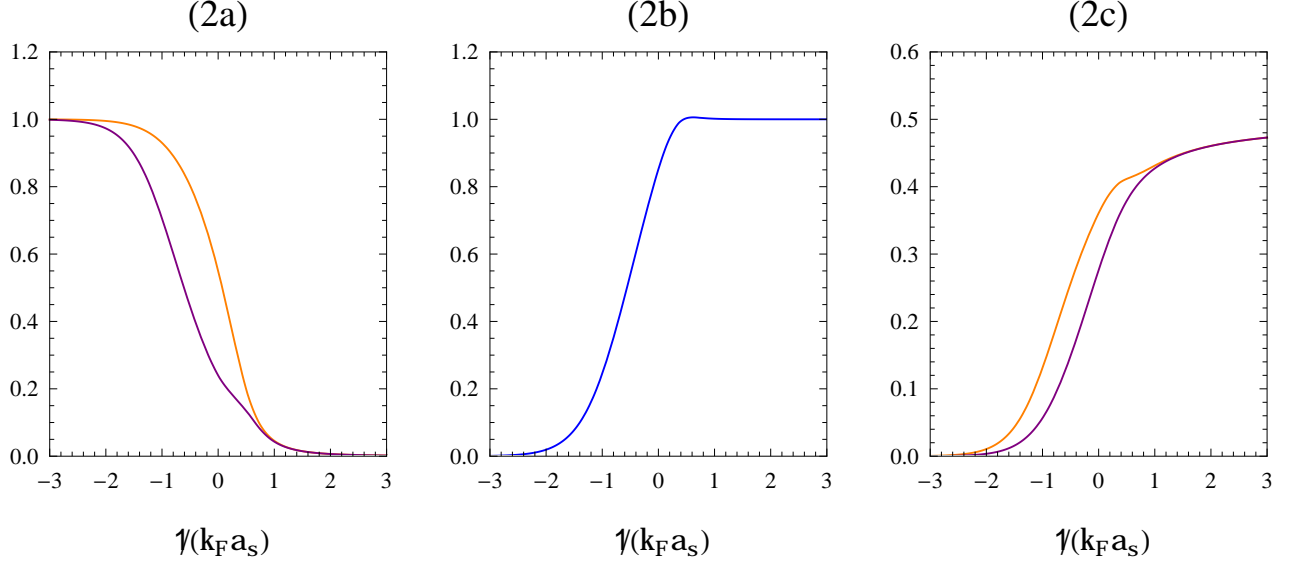


FIG. 2: The plots of (a): $N_0/(mk_F/2\pi^2)$ (orange) and $\eta/(mk_F/2\pi^2)$ (purple); (b): α/ρ_0 , and (c): κ/ρ_0 (orange) and $|\phi_0|^2/\rho_0$ (purple), as a function of $1/k_F a_S$ with a given $\bar{g} = 0.9$. All are independent of u_B .

where

$$\zeta = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \left[\frac{1}{8E_p^3} \left[\left(1 - 3 \frac{g^2 |\phi_0|^2}{E_p^2} \right) \frac{\varepsilon_p}{m} + \left(5 \frac{g^2 |\phi_0|^2}{E_p^2} \left(1 - \frac{g^2 |\phi_0|^2}{E_p^2} \right) \right) \frac{|\mathbf{p}|^2 (\hat{\mathbf{p}} \cdot \hat{\nabla})^2}{m^2} \right] \right]. \quad (67)$$

In (67) $\hat{\mathbf{p}}$ and $\hat{\nabla}$ are the unit vectors along the direction \mathbf{p} and the direction of the spatial variation of the phase mode θ respectively.

In particular, the chemical potential in the deep BCS regime is given by $\mu = \epsilon_F$, and then turns to be negative with the value $|\mu| \simeq 1/2ma_S^2 \rightarrow \infty$ in the deep BEC regime. The mode-coupling coefficient α vanishes in the deep BCS regime resulting from the particle-hole symmetry. At large values of $1/a_S k_F$ in the BEC regime, the scale factor κ is dominated by $\kappa \approx \rho_0/2$ so that $\alpha \rightarrow \rho_0$. In the deep BCS regime N_0 is the density of states at the Fermi surface, namely $N_0 = mk_F/2\pi^2$. Then in this regime the scale factor κ is approximated by $\kappa \simeq \rho_0/4mg^2\zeta$ in that $\zeta \approx k_F \epsilon_F / 18\pi^2 g^2 |\phi_0|^2$. With the approximated results of κ and ζ , the behaviour of η in the deep BCS regime is simply given by $\eta \simeq 3\rho_0/4\epsilon_F = mk_F/2\pi^2$ when $\rho_0 \equiv k_F^3/3\pi^2$ is defined. Thus, $N_0 \approx \eta$ in the deep BCS regime. In the BEC regime both N_0 and $\eta_0 \approx N_0 \simeq g^2 |\phi_0|^2 (2m|\mu|)^{3/2}/|\mu|^3$ vanish due to $|\mu| \rightarrow \infty$, with $N_0/\eta \rightarrow 1$, again independently of u_B . It is convenient to work with the dimensionless couplings \bar{g} and \bar{u}_B , defined by $\bar{g}^2 = (3k_F^3/64\epsilon_F^2)g^2$ and $\bar{u}_B = (3/64)(\epsilon_F/k_F)u_B$. See Fig.2.

Finally, in the definition of \bar{M}^2 , we have used the relationship between the s -wave scattering length a_S and the binding energy in (5). In the deep BEC regime \bar{M}_0 vanishes as N_0 such that $\bar{M}_0^2/N_0 \rightarrow 2g^2\rho_0$.

Numerical values

To be concrete, consider the resonance in ${}^6\text{Li}$ at $H_0 = 543.25G$, discussed in some detail in [21]. [This is to be distinguished from the very broad Feshbach resonance in ${}^6\text{Li}$ at 850 G.] As our benchmark we take the number density $\rho_0 \approx 3 \times 10^{12} \text{cm}^{-3}$, whence $\epsilon_F \approx 7 \times 10^{-11} \text{eV}$. Figures 2 give (dimensionless) values of N_0 and η , α and κ and $|\phi_0|^2$ respectively with asymptotic values (1,0) for Fig. 2a, (0,1) for Fig. 2b and (0, 0.5) for Fig. 2c. In Fig.3 we display the u_B -dependent \bar{M}^2 and c^2 for a range of \bar{u}_B .

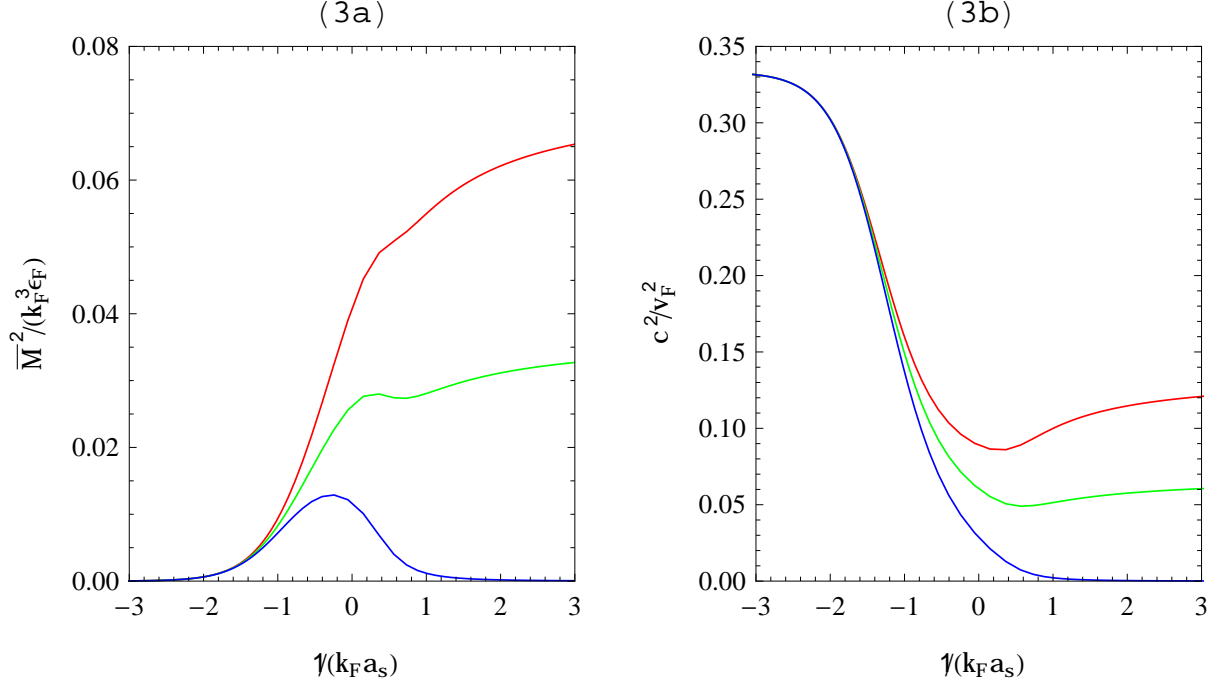


FIG. 3: Fig.3 The plots of (a) $\bar{M}^2/(k_F^3 \epsilon_F)$, and (b) c^2/v_F^2 , as a function of $1/k_F a_S$ for the value $\bar{g} = 0.9$ and $\bar{u}_B = 3$ (red), 1.5(green), 0(blue). In the BCS regime $c^2/v_F^2 \rightarrow 1/3$.

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